Quantum Process Tomography of a Generalized Pauli Channel

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Abstract— **A quantum channel tomography method is worked out for the family of so-called Pauli channels. The results are given for 2-level Pauli channels and then solved for generalized Pauli channels also.**

I. INTRODUCTION

In the quantum mechanical area, both dynamical changes and communication is treated using *quantum channels*. They are nothing else but trace preserving completely positive mappings $\mathcal{E}$ which transform the input state ρ given on the input of the channel to the output state $\mathcal{E}(\rho)$ appearing on the other side. A reasonable assumption that the channel belongs to a parametric family, i.e for fixed input state the output states belongs to the parametric family $\{\mathcal{E}_\theta(\rho)\}_{\theta\in\Theta}$, hence the channel estimation problem can be traced back to parameter estimation problem.

It is a fundamental problem of quantum information theory to estimate the parameters of a channel, because quantum communication usually requires a priori knowledge of the properties of the channel. The estimation methods are using a known input state and measurement data from the output to the estimation, so our aim is to determine optimal input state and optimal parameter estimator [1]. It is not very long since the quantum channel identification problem was directed proper attention, and the theory of finding an optimal estimation scheme has been investigated so far only in few papers [2], [3], [4].

This paper aims quantum channel identification, namely the identification of a generalized Pauli-channel. Generalized Pauli channels form a large class of quantum channels, that's the reason for choosing it as a subject of our work. A former work in this topic is introduced in [5], but we are using in our paper a different, more natural generalization of the Pauli channels.

The cornerstones of channel parameter estimation procedure are the *known input state* which can be manipulated by the user in order to obtain better performance; *the measurement strategy* which determines the way of information extraction from the quantum state appearing on the output. In this work, we are using a *mutually unbiased bases* [6] based measurement strategy which is a generalization the so-called *Pauli-spin measurement* used in 2-level systems [7] . The performance indicator of an estimation scheme is usually the *mean squared error matrix* (which is simply the covariance matrix in the case of an unbiased estimator). In order to have a good estimator we are minimizing the trace of the MSE matrix.

The paper is organized as follows. In Section II, the 2-level quantum channel estimation problem is detailed and solved. Afterwards, in Section III, the general (i.e. n-level) Pauli channel estimation problem is formulated and solved. Finally, in Section IV some concluding remarks are given. A few possible directions of the future work are also given here.

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II. THE QUBIT CASE

The simplest type of the Pauli channels occurs in the case of 2-level quantum systems, i.e. when the input and output states of the channel are both quantum bits (qubits). This means, that the quantum channel is a mapping from a 2 dimensional Hilbert space $\mathcal{H}_1$ to another 2 dimensional Hilbert space $\mathcal{H}_2$.

A. Notations

The qubits can be described according to the Bloch-parametrization as a vector inside the 3 dimensional unitary ball:

$$\theta = [x, y, z], \quad \text{where } x^2 + y^2 + z^2 \leq 1$$

The Pauli channel is determined by 3 direction vectors ($\underline{e}, \underline{f}, \underline{g}$) and 3 scalar parameters (λ, μ, ν), where the vectors are of unit length and they are orthogonal to each other, and the scalars are with less absolute value then 1.

If the input state can be written in the following form:

$$\theta = \alpha\underline{e} + \beta\underline{f} + \gamma\underline{g},$$

then the output state will be:

$$\mathcal{E}(\theta) = \lambda\alpha\underline{e} + \mu\beta\underline{f} + \nu\gamma\underline{g}.$$

Because of the symmetry of SO(3) we can assume that the vectors of the Pauli channel are in the directions of axes and the scalars are positive.

Hence in this scenario:

$$\alpha = x,\ \beta = y,\ \gamma = z,$$

and so

$$\mathcal{E}([x, y, z]) = [\lambda x, \mu y, \nu z] \tag{1}$$

B. Parameter estimation

The quantum measurement strategy we are using is the following one:
Take an input qubit being in the state $[x, y, z]$, send it through the Pauli channel, and than perform *standard measurements* [6]. By standard measurement we mean the measurement of the so-called *Pauli observables*. Practically, this means, that we are measuring the length of the Bloch vector components in the coordinate frame directions.

For example we will perform a measurement in the x direction accordingly to 1 on λx. The outcome Ψ_x is unbiased to λx and its variance is $1 - \lambda^2 x^2$. So an *unbiased estimator* for λ will be:

$$\hat{\lambda} = \frac{\Psi_x}{x}.$$

The variance of this estimator is

$$Var(\hat{\lambda}) = \frac{1 - \lambda^2 x^2}{x^2} = \frac{1}{x^2} - \lambda^2. \tag{2}$$

It is possible to estimate the remaining parameters with similar tools and results as in (2):

$$Var(\hat{\mu}) = \frac{1}{y^2} - \mu^2 \tag{3}$$

and

$$Var(\hat{\nu}) = \frac{1}{z^2} - \nu^2. \tag{4}$$

Let we take the Fisher information for λ:

$$I = \sum_{\xi=\{-1,1\}} p_\xi(x, \lambda) \left[\frac{d}{d\lambda} log\ p_\xi(x, \lambda) \right]^2 = \frac{x^2}{1 - \lambda^2 x^2} \tag{5}$$

According to the Cramer-Rao inequality

$$Var(\tilde{\lambda}) \geq I^{-1} = \frac{1 - \lambda^2 x^2}{x^2}$$

The inequality is sharp in the Equation (2), so these estimators (2)-(4) are *efficient*.

EXAMPLE 1: *Let the input state be* $\frac{1}{\sqrt{3}}[1, 1, 1]$.
Then the variances for the parameter estimation:

$$Var(\hat{\lambda}) = 3 - \lambda^2$$
$$Var(\hat{\mu}) = 3 - \mu^2$$
$$Var(\hat{\nu}) = 3 - \nu^2$$

EXAMPLE 2: *Input state:* $[1, 0, 0]$.
Then the variances for the parameter estimation:

$$Var(\hat{\lambda}) = 1 - \lambda^2$$
$$Var(\hat{\mu}) = Var(\hat{\nu}) = \infty.$$

As it can be seen, it is possible to find a much better estimation on λ but then it is impossible to obtain an estimate of the other two parameters.

So if we are allowed to use only one kind of input states, we want to minimize the sum

$$Var(\hat{\lambda})+Var(\hat{\mu})+Var(\hat{\nu}) = \frac{1}{x^2} - \lambda^2 + \frac{1}{y^2} - \mu^2 + \frac{1}{z^2} - \nu^2.$$

In this form λ, μ and ν are constant it is enough to minimize

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}$$

with respect to $x^2 + y^2 + z^2 \leq 1$.

Obviously minima is taken at the sphere of the Bloch-ball, else we increase one of the input coordinates and the variance will decrease. Hence the condition can be written in the form of $x^2 + y^2 + z^2 = 1$. Solving these minimization problem we get that the optimum is at

$$x = y = z = \frac{1}{\sqrt{3}},$$

if we take the nonnegative solution.

C. Multiple input case

In this section we will assume that we can have different input states. Because the number of parameters is 3, it is enough to have 3 different input states accordingly to the method exposed in the previous section.

Let be the input states

$$[x1, y1, z1],\ [x2, y2, z2] \text{ and } [x3, y3, z3].$$

In this case we can get three different parameter estimation from three different input states:

$$Var(\hat{\lambda}_1) = \frac{1}{x_1^2} - \lambda^2$$
$$Var(\hat{\lambda}_2) = \frac{1}{x_2^2} - \lambda^2$$
$$Var(\hat{\lambda}_3) = \frac{1}{x_3^2} - \lambda^2$$
$$Var(\hat{\mu}_1) = \frac{1}{y_1^2} - \mu^2$$
$$\cdots$$
$$Var(\hat{\nu}_3) = \frac{1}{z_3^2} - \nu^2$$

We want to construct one estimator from 3 different estimators.

1) Arithmetical mean: The simplest way is to take the arithmetical mean of different estimators. So in this case

$$\hat{\lambda} = \frac{\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3}{3}$$

and so

$$Var(\hat{\lambda}) = \frac{1}{9} \left[Var(\hat{\lambda}_1) + Var(\hat{\lambda}_2) + Var(\hat{\lambda}_3) \right]$$

and similar formulas stands for the estimators of μ and ν.

So the task is to minimize

$$\frac{1}{9} \left[\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} - 3\lambda^2 \right] +$$
$$+ \frac{1}{9} \left[\frac{1}{y_1^2} + \frac{1}{y_2^2} + \frac{1}{y_3^2} - 3\mu^2 \right] +$$

$$+\frac{1}{9}\left[\frac{1}{z_1^2}+\frac{1}{z_2^2}+\frac{1}{z_3^2}-3\nu^2\right]$$

with respect to $x_1^2+y_1^2+z_1^2 \leq 1$, $x_2^2+y_2^2+z_2^2 \leq 1$ and $x_3^2+y_3^2+z_3^2 \leq 1$.

Solving this optimization problem we get the same as in previous section: the unique nonnegative solution is

$$[x1, y1, z1] = [x2, y2, z2] = [x3, y3, z3] = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$$

The optimal variance is:

$$3 - \frac{\lambda^2}{3} - \frac{\mu^2}{3} - \frac{\nu^2}{3}$$

2) Convex combination: More complex is the problem, if we take the convex combination of the different estimators:

$$\hat{\lambda} = A\hat{\lambda}_1 + B\hat{\lambda}_2 + C\hat{\lambda}_3,$$

where A, B, C are nonnegative and $A + B + C = 1$.

Easily comes that the optimal ratios are the following:

$$A : B : C = \frac{1}{Var(\hat{\lambda}_1)} : \frac{1}{Var(\hat{\lambda}_2)} : \frac{1}{Var(\hat{\lambda}_3)}$$

So in the optimal case

$$A = \frac{\frac{1}{Var(\hat{\lambda}_1)}}{\frac{1}{Var(\hat{\lambda}_1)} + \frac{1}{Var(\hat{\lambda}_2)} + \frac{1}{Var(\hat{\lambda}_3)}},$$

and similarly goes for the other cases, hence using some computation comes that the optimal variance is

$$Var(\hat{\lambda}_{opt}) = \frac{1}{\frac{1}{Var(\hat{\lambda}_1)} + \frac{1}{Var(\hat{\lambda}_2)} + \frac{1}{Var(\hat{\lambda}_3)}}$$

In this situation this is exactly

$$Var(\hat{\lambda}_{opt}) = \frac{1}{\frac{1}{1/x_1^2-\lambda^2} + \frac{1}{1/x_2^2-\lambda^2} + \frac{1}{1/x_3^2-\lambda^2}}.$$

And similar results are true for $Var(\hat{\mu}_{opt})$ and $Var(\hat{\nu}_{opt})$.

So our task is to minimize

$$\frac{1}{\frac{1}{1/x_1^2-\lambda^2} + \frac{1}{1/x_2^2-\lambda^2} + \frac{1}{1/x_3^2-\lambda^2}} +$$
$$+\frac{1}{\frac{1}{1/y_1^2-\mu^2} + \frac{1}{1/y_2^2-\mu^2} + \frac{1}{1/y_3^2-\mu^2}} +$$
$$+\frac{1}{\frac{1}{1/z_1^2-\nu^2} + \frac{1}{1/z_2^2-\nu^2} + \frac{1}{1/z_3^2-\nu^2}}$$

with respect to $x_1^2+y_1^2+z_1^2 \leq 1$, $x_2^2+y_2^2+z_2^2 \leq 1$ and $x_3^2+y_3^2+z_3^2 \leq 1$.

If we solve this problem, we get that there is essentially an unique optimum:

$$[x1, y1, z1] = [1, 0, 0]$$

$$[x2, y2, z2] = [0, 1, 0]$$

$$[x3, y3, z3] = [0, 0, 1]$$

The optimal variance is

$$3 - \lambda^2 - \mu^2 - \nu^2$$

This is obviously less than by the arithmetical mean, because that is a special case of this. We can see that we can get more accurate estimation if we estimate the parameters separately, than if we maximize information only in one step.

Accordingly to (5) the Fisher information of three measurements in the direction of x are:

$$I_1 = \frac{x_1^2}{1-\lambda^2 x_1^2} \quad, I_2 = \frac{x_2^2}{1-\lambda^2 x_2^2} \quad \text{and } I_3 = \frac{x_3^2}{1-\lambda^2 x_3^2}$$

The measurements are independent so the cumulated Fisher information is simply the sum of these:

$$I = I_1 + I_2 + I_3$$

and we can check that

$$Var(\hat{\lambda}) = I^{-1}$$

and that imply the efficiency of estimator.

An interest thing that because of rotation symmetry, we get the same result, if the direction of channel ($\underline{e}, \underline{f}, \underline{g}$) is not in the direction of axes, i.e. then the optimal input states will be $\underline{e}$,$\underline{f}$ and $\underline{g}$.

Moreover, if we do not know the direction of the channel the same thoughts are valid: Suppose, that directions $\underline{e}, \underline{f}, \underline{g}$ are known, then the optimal input states are also known. This means, that the algorithm has two steps.

In the first step the channel directions are estimated - the methods is not relevant, the only thing that matters that if the measurement numbers goes to infinity the estimation converges to the real directions.

Afterwards, as a second step an estimation is computed using the input states of the previously estimated directions. If the first step tends to infinity but in such way that it is be negligible to the second step (for example the length of first part is $\sqrt{N}$ when the total number of measurement is N) then the algorithm is asymptotically optimal.

III. GENERALIZED PAULI CHANNELS

A. Representation of quantum states using mutually unbiased bases

In what follows, the three main elements of the quantum process estimation scheme is detailed. The important questions are what kind of quantum states are sent through the generalized Pauli channel, how does the channel works, and how it is parameterized, and finally, what type of state tomography method is used to estimate the output of the channel.

Since one of our our aims is to find the optimal input qubit state to a quantum channel, the state is not fixed. In the general case, a generalization of the Bloch parametrization is used in terms of *mutually unbiased bases*.

Suppose, that $r = n + 1$, and

$$f_1^i, f_2^i, \ldots, f_n^i \qquad (1 \leq i \leq r)$$

are mutually unbiased bases, i.e. $|\langle f_k^i, f_l^j\rangle| = 1/\sqrt{n}$ when $i \neq j$. The trace preserving conditional expectation E_i onto the subalgebra $\mathcal{A}_i$ of operators diagonal in the ith basis is

$$E_i(A) = \sum_{j=1}^{n} \langle f_j^i, A, f_j^i\rangle |f_j^i\rangle\langle f_j^i| \qquad (1 \le i \le r).$$

It is known from [6], that a statistical operator ρ can be expanded in the form

$$\rho = -I + \sum_{i=1}^{r} E_i(\rho)\,.$$

This formula can be used to parameterize the statistical operators as

$$E_i(\rho) = \sum_{j=1}^{n} y_j^i \cdot |f_j^i\rangle\langle f_j^i|, \quad i = 1, \ldots, r. \tag{6}$$

The parameters y_j^i satisfy the conditions

$$y_j^i \ge 0 \quad \text{and} \quad \sum_{j=1}^{n} y_j^i = 1. \tag{7}$$

The MUB is constructed as follows [6]. Let $e_0, e_1, \ldots, e_{n-1}$ be a basis, and let X be a unitary operator such that

$$Xe_i = \begin{cases} e_{i+1} & \text{if } \ 0 \le i \le n-2 \\ e_0 & \text{if } \ i = n-1 \end{cases}$$

Let Z be another unitary operator such that

$$Ze_k = e^{\mathrm{i}k2\pi/n} e_k.$$

The unitaries $S_{jk} := Z^j X^k$ $(0 \le j, k \le n-1)$ are pairwise orthogonal with respect to the Hilbert-Schmidt inner product. From the eigenvectors we can get MUBs. It is necessary to construct $n+1$ Matrices using the above formula. The first two are naturally X and Z, the remaining $n-1$ will be in the form ZX^k, $k = 1, \ldots, n-1$.

The MUB vectors (f_j^i) are the eigenvectors of the above $n+1$ matrices. Note, that the above choice of the $n+1$ mutually unbiased bases is arbitrary, it is possible to choose a different group of bases. This one was preferred because it was easy to algorithmize its generation.

B. *Generalized Pauli channel $\mathcal{E}$*

Instead of the simple *Pauli channel*, which has three real parameters, and works as follows

$$\mathcal{F}(\rho) = \frac{1}{2}\left(I + \lambda_1\theta_1\sigma_1 + \lambda_2\theta_2\sigma_2 + \lambda_3\theta_3\sigma_3\right),$$

when

$$\rho = \frac{1}{2}\left(I + \theta_1\sigma_1 + \theta_2\sigma_2 + \theta_3\sigma_3\right),$$

we are using a generalized Pauli channel $\mathcal{E}$ [7], which works as follows:

$$\mathcal{E}(\rho) = \left(1 - \sum_{i=1}^{r} \lambda_i\right) \frac{\operatorname{Tr}\rho}{n} I + \sum_{i=1}^{r} \lambda_i E_i(\rho),$$

where the parameters of the channel satisfy the inequality [7]

$$1 + n\lambda_j \ge \sum_j \lambda_j \ge -\frac{1}{n-1}. \tag{8}$$

C. *Channel estimation as a state tomography problem*

Unfortunately, the MUB operators are self-adjoint only in the 2 level case, i.e. in general, it is not possible to use them as observables (as they may have complex eigenvalues).

However, it is possible to construct observables of the form

$$A_i = \sum_{j=1}^{n} c_j^i \cdot |f_j^i\rangle\langle f_j^i|, \quad i = 1, \ldots, r, \tag{9}$$

where $c_j^i \in \mathbb{R}$, $\ j = 1, \ldots, n$ in the general case. This way, one gets observables in the n dimensional Hilbert space with eigenvalues c_1^i, c_2^i, etc. c_n^i. Note, that the actual values of the eigenvalues are practically irrelevant as the estimator is computed from the *probability* of the outcome and not the outcome itself. Since the task is to estimate the r parameters of the generalized Pauli channel, it is enough to use r of them.

The expected values of such observables are in the form $\operatorname{Tr}(\mathcal{E}(\rho)A_i)$. More generally, we have:

$$\begin{aligned}
\operatorname{Prob}(A_i = c_j^i) &= \operatorname{Tr}\mathcal{E}(\rho)|f_j^i\rangle\langle f_j^i| \\
&= \tfrac{1}{n}\Big(1 - \sum_j \lambda_j\Big)\operatorname{Tr}|f_j^i\rangle\langle f_j^i| + \\
&\quad + \sum_{k=1}^{n} \lambda_k \operatorname{Tr}\mathcal{E}_k(\rho)|f_j^i\rangle\langle f_j^i| \\
&= \tfrac{1}{n}\Big(1 - \sum_j \lambda_j\Big) + \lambda_i y_j^i + \tfrac{1}{n}\sum_{k \neq i} \lambda_k \\
&= \tfrac{1}{n} + \lambda_i\Big(y_j^i - \tfrac{1}{n}\Big)
\end{aligned}$$

It is easy to see, that the probabilities of the different outcomes of the observables depend only on the actual λ_i and the y_j^i coefficient of the projectors in (6). From the above, it is possible to derive n equivalent formulas for the estimator $\hat{\lambda}_i$:

$$\hat{\lambda}_i = \frac{n\nu(k, A_i, c_j^i) - 1}{ny_j^i - 1}, \quad j = 1, \ldots, n, \tag{10}$$

where $\nu(k, A_i, c_j^i)$ denotes the relative frequency of the c_j^i outcomes of the measurement of A_i after k measurements.

It is possible to use any group of the above estimators (supposed, that we have one from each group) to estimate the parameters of the generalized Pauli channel *componentwise*. Moreover, any of the above estimators give an *unbiased estimate*, since their expected values are $\lambda_1, \ldots \lambda_r$, respectively.

It is worth to think about a unified estimator for each λ_i which is some function of estimators (10). The simplest choice might be the arithmetical mean of them:

$$\bar{\lambda}_i = \frac{1}{n}\sum_{j=1}^{n} \frac{n\nu(k, A_i, c_j^i) - 1}{ny_j^i - 1}$$

EXAMPLE 3: *Three level Example*
In the three level case, the observables (9) are the following ones:

$$\begin{bmatrix} \frac{c_1^1+c_2^1+c_3^1}{3} & \frac{2c_1^1-c_2^1-c_3^1}{6}+\mathrm{i}\frac{c_2^1-c_3^1}{2\sqrt{3}} & \frac{2c_1^1-c_2^1-c_3^1}{6}-\mathrm{i}\frac{c_2^1-c_3^1}{2\sqrt{3}} \\ \frac{2c_1^1-c_2^1-c_3^1}{6}-\mathrm{i}\frac{c_2^1-c_3^1}{2\sqrt{3}} & \frac{c_1^1+c_2^1+c_3^1}{3} & \frac{2c_1^1-c_2^1-2c_3^1}{6}+\mathrm{i}\frac{c_2^1-c_3^1}{2\sqrt{3}} \\ \frac{2c_1^1-c_2^1-c_3^1}{6}+\mathrm{i}\frac{c_2^1-c_3^1}{2\sqrt{3}} & \frac{2c_1^1-c_2^1-c_3^1}{6}-\mathrm{i}\frac{c_2^1-c_3^1}{2\sqrt{3}} & \frac{c_1^1+c_2^1+c_3^1}{3} \end{bmatrix}$$

$$\begin{bmatrix} c_2^2 & 0 & 0 \\ 0 & c_1^2 & 0 \\ 0 & 0 & c_3^2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{c_1^3+c_2^3+c_3^3}{3} & \frac{2c_1^3-c_2^3-c_3^3}{6}+\mathrm{i}\frac{c_2^3-c_3^3}{2\sqrt{3}} & \frac{-c_1^3-c_2^3+2c_3^3}{6}-\mathrm{i}\frac{c_1^3-c_2^3}{2\sqrt{3}} \\ \frac{2c_1^3-c_2^3-c_3^3}{6}-\mathrm{i}\frac{c_2^3-c_3^3}{2\sqrt{3}} & \frac{c_1^3+c_2^3+c_3^3}{3} & \frac{-c_1^3+2c_2^3-c_3^3}{6}-\mathrm{i}\frac{c_1^3-c_3^3}{2\sqrt{3}} \\ \frac{-c_1^3-c_2^3+2c_3^3}{6}+\mathrm{i}\frac{c_1^3-c_2^3}{2\sqrt{3}} & \frac{-c_1^3+2c_2^3-c_3^3}{6}+\mathrm{i}\frac{c_1^3-c_3^3}{2\sqrt{3}} & \frac{c_1^3+c_2^3+c_3^3}{3} \end{bmatrix}$$

$$\begin{bmatrix} \frac{c_1^4+c_2^4+c_3^4}{3} & \frac{-c_1^4-c_2^4+2c_3^4}{6}-\mathrm{i}\frac{c_1^4-c_2^4}{2\sqrt{3}} & \frac{-c_1^4+2c_2^4-c_3^4}{6}-\mathrm{i}\frac{c_1^4-c_3^4}{2\sqrt{3}} \\ \frac{-c_1^4-c_2^4+2c_3^4}{6}+\mathrm{i}\frac{c_1^4-c_2^4}{2\sqrt{3}} & \frac{c_1^4+c_2^4+c_3^4}{3} & \frac{2c_1^4-c_2^4-c_3^4}{6}-\mathrm{i}\frac{c_2^4-c_3^4}{2\sqrt{3}} \\ \frac{-c_1^4+2c_2^4-c_3^4}{6}+\mathrm{i}\frac{c_1^4-c_3^4}{2\sqrt{3}} & \frac{2c_1^4-c_2^4-c_3^4}{6}+\mathrm{i}\frac{c_2^4-c_3^4}{2\sqrt{3}} & \frac{c_1^4+c_2^4+c_3^4}{3} \end{bmatrix}$$

D. *Optimization of the input state*

Computing the mean squared error matrix (MSE) based on the estimator (10), it will be a $r \times r$ symmetric matrix, moreover, because of the independency of the estimators, it will be diagonal:

$$V(\lambda_1, \dots, \lambda_r) = \begin{bmatrix} V_{11} & 0 & \dots & 0 \\ 0 & V_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & V_{rr} \end{bmatrix},$$

where the diagonal entries are the following expressions

$$V_{ii} = -\lambda_i^2 - \frac{(n-2)\lambda_i}{1-ny_j^i} + \frac{n-1}{(1-ny_j^i)^2}, \quad i = 1, \dots, r.$$

(Keep in mind, that n denotes the dimension of the Hilbert-space, and not the number of measurements performed.) Thus, the problem is to find a quantum state which minimizes V, i.e. makes the estimation the most precise. Since the matrix is diagonal, it is enough to minimize $\mathrm{Tr}\,(V)$, so we are searching for a density matrix ρ_{opt} (parameterized by $y_j^i, \quad j = 1, \dots, n, \quad i = 1, \dots, r,$) for which

$$\begin{aligned} \rho_{\text{opt}} \quad = \quad & \operatorname{argmin}_\rho \mathrm{Tr}\,(V) = \operatorname{argmin}_\rho \textstyle\sum_{i=1}^r V_{ii} \\ & \text{s.t. } \rho > 0, \quad \mathrm{Tr}\,(\rho) = 1, \quad \rho = \rho^* \end{aligned}$$

holds. Since there are only one y_j^i in each diagonal element of V, it is possible to minimize the trace of V component-wise, i.e. V_{ii} is minimized using y_j^i. (Index j coincides with the used estimator among the n equivalent ones in (10).) The first derivative of $\mathrm{Tr}\,(V)$ with respect to y_j^i is

$$\frac{\mathrm{d}\ \mathrm{Tr}\,(V)}{\mathrm{d}\ y_j^i} = \frac{\mathrm{d}\ V_{ii}}{\mathrm{d}\ y_j^i} = -\frac{(n-2)n\lambda_i}{(1-ny_j^i)^2} + \frac{2(n-1)n}{(1-ny_j^i)^3}$$

The derivative is 0 at

$$\tilde{y}_j^i = \frac{\lambda_1(n-2) - 2n + 2}{\lambda_1 n(n-2)} \tag{11}$$

Substituting $\tilde{y}_j^i$ to the second derivative:

$$\frac{\mathrm{d}^2\ V_{ii}}{\mathrm{d}\ t^2}(\tilde{y}_j^i) = \frac{(n-2)^4 n^2 \lambda_1^4}{8(n-1)^3} > 0,$$

i.e. the there is a minima at $\tilde{y}_j^i$. The optimal input state must be determined by combining (11) with constraints (7) and (8).

IV. CONCLUSIONS AND FUTURE WORKS

In this work, a quantum Pauli channel estimation framework has been worked out. As a first step, the 2 level channel case was investigated. It have been shown that the naive approach is not optimal, different input states are necessary for optimal estimation. Moreover it has been proven that the optimal input states are the Pauli states, and standard measurement scheme provides optimal estimates of the parameters.

In Section III the general case is discussed. We are introducing mutually unbiased bases to represent higher dimensional quantum states. After the description of the general Pauli channel, the estimators for the channel parameters has been given. Since the problem is very complicated, we are simply using the mean of different estimators to get analytical results. Finally the optimal input states, which minimizes the trace of mean squared error matrix for parameters has been determined.

A future work would be a deeper examination of the restrictions on the input states and on the channel parameters.

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